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Mass Points of Measures and Orthogonal Polynomials on the Unit Circle[1]

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Orthogonal polynomials on the unit circle are completely determined by their reflection coefficients through the Szegő recurrences. We assume that the reflection coefficients tend to some complex number a with $0<|a|<1$. The orthogonality measure μ then lives essentially on the arc $\{e^{it}: \alpha \leqslant t \leqslant 2\pi - \alpha\}$ where $\sin\frac{\alpha}{2} \stackrel{\text{def}}{=} |a|$ with $\alpha \in (0, \pi)$. Under the certain rate of convergence it was proved in (Golinskii *et al.* (*J. Approx. Theory* **96** (1999), 1–32)) that μ has no mass points inside this arc. We show that this result is sharp in a sense. We also examine the case of the whole unit circle and some examples of singular continuous measures given by their reflection coefficients. © 2002 Elsevier Science (USA)

Key Words: measures on the unit circle; orthogonal polynomials; reflection coefficients; transfer matrices.

1. INTRODUCTION

Let μ be a probability measure on the unit circle $\mathbb{T} = \{|\zeta| = 1\}$ with the infinite support. The latter is defined as the smallest *closed* set with the complement having μ-measure zero. The polynomials $\varphi_n(z) = \varphi_n(\mu, z) = \kappa_n(\mu) z^n + \cdots$, orthonormal on the unit circle with respect to μ are uniquely determined by the requirement that $\kappa_n = \kappa_n(\mu) > 0$ and

$$\int_{\mathbb{T}} \varphi_n(\zeta)\overline{\varphi_m(\zeta)}\, d\mu = \delta_{n,m}, \qquad n, m = 0, 1, \ldots, \quad \zeta \in \mathbb{T}. \tag{1}$$

It is well known (see, e.g., [5]) that the theory of orthogonal polynomials on the unit circle can be viewed as a theory of the first-order vector

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difference equation

$$\vec{X}(z,n) = T(z,a_n)\vec{X}(z,n-1), \qquad n \in \mathbb{N} \stackrel{\text{def}}{=} \{1,2,\ldots\},$$

where

$$T(z,a_n) \stackrel{\text{def}}{=} \frac{1}{\rho_n}\begin{pmatrix} z & a_n \\ \bar{a}_n z & 1 \end{pmatrix}, \qquad \rho_n^2 \stackrel{\text{def}}{=} 1 - |a_n|^2, \tag{2}$$

and $\{a_n\}$ is an arbitrary sequence of complex numbers with $|a_n| < 1$. This equation is called the Szegő equation and T the Szegő matrix. The relation

$$\begin{bmatrix} \varphi_n(z) \\ \varphi_n^*(z) \end{bmatrix} = T(z,a_n)\begin{bmatrix} \varphi_{n-1}(z) \\ \varphi_{n-1}^*(z) \end{bmatrix}, \qquad n \geqslant 1, \quad \varphi_0 = \varphi_0^* = 1 \tag{3}$$

is just a vector form of the known Szegő recurrences (cf. [12 formula (11.4.7)]). Here, the reversed $*$-polynomial of a polynomial p_n of degree n is defined by $p_n^*(z) \stackrel{\text{def}}{=} z^n \overline{p_n(1/\bar{z})}$. In the orthogonal polynomials setting the numbers a_n are known as the *reflection coefficients* and $a_n = \kappa_n^{-1}\varphi_n(0)$. The Favard theorem for the unit circle states that each sequence $\{a_n\}$ from the open unit disk $\mathbb{D}$ comes up as a sequence of reflection coefficients for a certain uniquely determined probability measure μ.

Let $\mathscr{T}_s(z) \stackrel{\text{def}}{=} T(z,a_s)T(z,a_{s-1})\cdots T(z,a_1)$, $s \geqslant 1$, be the transfer matrix. Then (3) can be written as

$$\begin{bmatrix} \varphi_s(z) \\ \varphi_s^*(z) \end{bmatrix} = \mathscr{T}_s(z)\begin{bmatrix} 1 \\ 1 \end{bmatrix}. \tag{4}$$

For each Szegő matrix $T(z,a)$ its eigenvalues $\{r_1, r_2\}$, i.e., the roots of the characteristic equation

$$r^2 - \frac{z+1}{\rho} r + z = 0, \qquad \rho^2 = 1 - |a|^2 \tag{5}$$

can be found explicitly (cf. [6, Sect. 2])

$$r_{1,2}(z) = \frac{z + 1 \pm \sqrt{(z-e^{i\alpha})(z-e^{-i\alpha})}}{2\rho}, \qquad \alpha \stackrel{\text{def}}{=} 2\arcsin|a|.$$

By the Vieta formulas

$$\rho(r_1 + r_2) = z + 1, \qquad r_1 r_2 = z. \tag{6}$$

There are two arcs of the unit circle pertaining to the number a:

$$\Delta_\alpha \stackrel{\text{def}}{=} \{e^{it} : \alpha \leqslant t \leqslant 2\pi - \alpha\}, \quad \Delta_\alpha^0 \stackrel{\text{def}}{=} \{e^{it} : \alpha < t < 2\pi - \alpha\}. \tag{7}$$

It is not hard to see that $|r_1(z)| = |r_2(z)| = 1$ for $z \in \Delta_\alpha$ (and $r_1 = r_2$ only at the endpoints of the arc), and $|r_1(z)| > 1 > |r_2(z)|$ off Δ_α. Moreover, for $z = e^{it} \in \Delta_\alpha$

$$z_{1,2}(e^{it}) \stackrel{\text{def}}{=} \rho r_{1,2}(e^{it}) = e^{i\frac{t}{2}}\Big(\cos\frac{t}{2} \pm ig(t)\Big), \qquad g(t) \stackrel{\text{def}}{=} \sqrt{\cos^2\frac{\alpha}{2} - \cos^2\frac{t}{2}}. \tag{8}$$

Let us point out that both eigenvalues r_j and arcs (7) are completely determined by the absolute value $|a|$ (and independent of the argument of a).

For $z \in \Delta_\alpha^0$ the Szegő matrix T can be reduced to diagonal form

$$T(z,a) = V(z)R(z)V^{-1}(z), \qquad R(z) \stackrel{\text{def}}{=} \begin{pmatrix} r_1(z) & 0 \\ 0 & r_2(z) \end{pmatrix}. \tag{9}$$

Here V may be taken as

$$V = \begin{pmatrix} 1 & 1 \\ \xi_1 & \xi_2 \end{pmatrix}, \qquad V^{-1} = \frac{1}{\xi_2 - \xi_1}\begin{pmatrix} \xi_2 & -1 \\ -\xi_1 & 1 \end{pmatrix},$$

where $\xi_j = \xi_j(z)$ are defined by the equalities

$$z + a\xi_j = \rho r_j = z_j, \qquad j = 1, 2. \tag{10}$$

The starting point for our investigation is the following result obtained in [6, Corollary 13, p. 21].

Theorem A. *Let* $\lim_{n\to\infty} a_n = a$, $0 < |a| < 1$ *and suppose that for every real* t

$$\sum_{n=1}^{\infty} \exp\left\{ t \sum_{k=1}^{n} |a_k - a| \right\} = \infty. \tag{11}$$

Then the corresponding orthogonality measure has no mass points in Δ_α^0.

Note that (11) holds whenever $|a_n - a| = o(1/n)$. Our goal is to show that this result is sharp in a way. Let $0 < |b| < 1$, denote by $\mathscr{B}(b)$ the set of all sequences $\{b_n\}$ of complex numbers which satisfy

(i) $|b_n| < 1$ for $n \geqslant 1$;

(ii) $b_n = b(1 + \varepsilon_n)$, $\lim_{n\to\infty} \varepsilon_n = 0$;

(iii) $\sum_{n=0}^{\infty} |\varepsilon_n| = \infty$.

DEFINITION. Two sequences $\{a'_n\} \in \mathscr{B}(a')$ and $\{a''_n\} \in \mathscr{B}(a'')$ are said to be *equivalent* if

$$\lim_{n\to\infty} \frac{\sum_{k=1}^{n} |\varepsilon'_k|}{\sum_{k=1}^{n} |\varepsilon''_k|} = 1.$$

It turns out that if (11) is false (that is, the series converges for some t), then there exists a sequence $\{a'_n\} \in \mathscr{B}(a')$, equivalent to the original one and such that the set of mass points for the corresponding orthogonality measure μ' is nonempty in an appropriate arc $\Delta^0_{\alpha'}$. The idea (we call it the "twisting–squeezing procedure") is adopted from [9], where the similar result about discrete Schrödinger operators is established. We complete the paper with relatively simple case of the whole unit circle and look at some examples of singular continuous measures given by their reflection coefficients.

2. TWISTING–SQUEEZING PROCEDURE

We will focus on the class of sample sequences of reflection coefficients, each of which is determined by the following triple $(a, \Lambda, \{\varepsilon_n\})$:

a is a nonzero complex number from $\mathbb{D}$;

$\Lambda = \{n_1 < n_2 < \cdots\}$ is a sequence of positive integers;

$\{\varepsilon_n\}_{n\geqslant 1}$ is a sequence of complex numbers with $|\varepsilon_n| < 1$, $\lim_{n\to\infty} \varepsilon_n = 0$ and $|1 + \varepsilon_n| = 1$.

Put

$$a_n = \begin{cases} a, & \text{for } n \notin \Lambda, \\ a(1 + \varepsilon_k) & \text{for } n = n_k. \end{cases} \tag{12}$$

Note that $|a_n| = |a|$ for all n.

The main objective of our paper is to show that the result in Theorem A is sharp in a sense.

THEOREM 1. *Let a sequence $\{a'_n\}$ of reflection coefficients satisfy $a'_n = a'(1 + \varepsilon'_n)$ with $\lim_{n\to\infty} \varepsilon'_n = 0$ and $0 < |a'| < 1$. Suppose that for some $M \in \mathbb{R}$*

$$\sum_{n=1}^{\infty} \exp\left\{ M \sum_{k=1}^{n} |a'_k - a'| \right\} = \sum_{n=1}^{\infty} \exp\left\{ M |a'| \sum_{k=1}^{n} |\varepsilon'_k| \right\} < \infty. \tag{13}$$

Then for an arbitrary $N \in \mathbb{N}$ there exists an equivalent sample sequence $\{a_n\}$ such that the set of mass points of the corresponding orthogonality measure μ on the arc Δ_α is nonempty and contains at least N points.

Write $T(\cdot, a) = T,\ T(\cdot, a_{n_k}) = T_k$. The transfer matrix $\mathcal{T}_s$ for such a sample sequence (12) takes the form

$$\mathcal{T}_s = T^{s-n_l} T_l T^{m_l-1} T_{l-1} T^{m_{l-1}-1} T_{l-2} \dots T_1 T^{m_1-1}, \qquad m_k = n_k - n_{k-1}, \tag{14}$$

where $l \in \mathbb{N},\ m_1 = n_1$ and $n_l \leqslant s < n_{l+1}$. Since $|a_n| = |a|$, the arc Δ_α^0 as well as the eigenvalues $r_{1,2}$ are *the same* for all Szegő matrices T and T_k in (14).

Let V reduce T to diagonal form on Δ_α^0 (see (9)). Then

$$\mathcal{T}_s = (VR^{s-n_l}V^{-1})T_l(VR^{m_l-1}V^{-1})T_{l-1} \dots T_1(VR^{m_1-1}V^{-1}).$$

If we slightly rearrange the factors, we come to the following representation for the transfer matrix:

$$\mathcal{T}_s = VR^{s-n_l+1}(R^{-1}V^{-1}T_lV)R^{m_l}(R^{-1}V^{-1}T_{l-1}V)\cdots(R^{-1}V^{-1}T_1V)R^{m_1}R^{-1}V^{-1}$$

$$= VR^{s-n_l+1} \overset{\curvearrowleft}{\prod_{1 \leqslant p \leqslant l}} A_p R^{m_p} \cdot R^{-1}V^{-1}, \qquad A_p \overset{\text{def}}{=} R^{-1}V^{-1}T_pV. \tag{15}$$

For $k = 1, 2, \dots, l$ consider the set of vectors

$$\begin{bmatrix} u_k \\ v_k \end{bmatrix} \overset{\text{def}}{=} A_k R^{m_k} \begin{bmatrix} u_{k-1} \\ v_{k-1} \end{bmatrix}, \qquad \begin{bmatrix} u_0 \\ v_0 \end{bmatrix} \overset{\text{def}}{=} R^{-1}V^{-1} \begin{bmatrix} 1 \\ 1 \end{bmatrix}. \tag{16}$$

Then

$$\begin{bmatrix} u_k \\ v_k \end{bmatrix} = \overset{\curvearrowleft}{\prod_{1 \leqslant j \leqslant k}} A_j R^{m_j} \begin{bmatrix} u_0 \\ v_0 \end{bmatrix}, \qquad \begin{bmatrix} \varphi_s \\ \varphi_s^* \end{bmatrix} = VR^{s-n_l+1} \begin{bmatrix} u_l \\ v_l \end{bmatrix}. \tag{17}$$

On the other hand, by the definition of A_k we have

$$\begin{bmatrix} u_k \\ v_k \end{bmatrix} = R^{-1}V^{-1} \overset{\curvearrowleft}{\prod_{1 \leqslant j \leqslant n_k}} T(z, a_j) \begin{bmatrix} 1 \\ 1 \end{bmatrix} = R^{-1}V^{-1} \begin{bmatrix} \varphi_{n_k} \\ \varphi_{n_k}^* \end{bmatrix}. \tag{18}$$

LEMMA 2. *Let* $0 < \alpha < \pi$. *For an arbitrary vector*

$$h = \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{C}^2, \qquad |x| = |y|$$

the following inequalities hold:

$$|\xi_1(z)x - y| \leqslant |\xi_2(z)x - y| \leqslant \cot\frac{\alpha}{4}|\xi_1(z)\,x - y|, \qquad z = e^{it} \in \Delta_\alpha^0,$$

where $\xi_{1,2}(z)$ *are defined in* (10).

Proof. Without loss of generality, we may assume that $|x| = |y| = |a|$, so that $x = ae^{it(x)}$, $y = ae^{it(y)}$, $0 \leqslant t(x), t(y) < 2\pi$. Put $t(x,y) = t(x) - t(y)$. Then

$$|\xi_2 x - y|^2 - |\xi_1 x - y|^2 = |a|^2(|\xi_2|^2 - |\xi_1|^2) - 2|a|^2 \Re\Big\{(\xi_2 - \xi_1)e^{it(x,y)}\Big\}.$$

For the first term we have

$$|a|^2(|\xi_2|^2 - |\xi_1|^2) = |z_2 - z|^2 - |z_1 - z|^2 = 2\Re\{z\overline{(z_1 - z_2)}\}.$$

By (8) $z\overline{(z_1 - z_2)} = -2ie^{it/2}\, g(t)$ and hence

$$|a|^2(|\xi_2|^2 - |\xi_1|^2) = 4\sin\frac{t}{2}\, g(t).$$

Next, $a(\xi_2 - \xi_1)e^{it(x,y)} = -2ig(t)\exp\{it/2 + it(x,y)\}$ so that

$$|\xi_2 x - y|^2 - |\xi_1 x - y|^2 = 4g(t)\left(\sin\frac{t}{2} + |a|^2 \Re\left\{\frac{i}{a} e^{i(t/2 + t(x,y))}\right\}\right). \quad (19)$$

Let us now calculate the sum

$$|\xi_2 x - y|^2 + |\xi_1 x - y|^2 = |a|^2(|\xi_2|^2 + |\xi_1|^2 + 2) - 2|a|^2 \Re\Big\{(\xi_2 + \xi_1)e^{it(x,y)}\Big\}.$$

Similarly, by using $z_1 + z_2 = z + 1$ (see (6)) we have

$$\begin{aligned} &|a|^2(|\xi_2|^2 + |\xi_1|^2 + 2) \\ &\quad = |z_2 - z|^2 + |z_1 - z|^2 + 2|a|^2 \\ &\quad = 2\rho^2 + 2 - 2\Re\{\bar{z}(z_1 + z_2)\} + 2|a|^2 = 4 - 2\Re\{\bar{z}(z+1)\} \\ &\quad = 2(1 - \Re z) = 4\sin^2\frac{t}{2}, \end{aligned}$$

$$a(\xi_1 + \xi_2)e^{it(x,y)} = (1 - z)e^{it(x,y)} \qquad \text{and} \qquad 2\Re\Big\{(\xi_1 + \xi_2)e^{it(x,y)}\Big\}$$
$$= -4\sin\frac{t}{2}\,\Re\left\{\frac{i}{a} e^{i(t/2 + t(x,y))}\right\}.$$

Finally,

$$|\xi_2 x - y|^2 + |\xi_1 x - y|^2 = 4\sin\frac{t}{2}\left(\sin\frac{t}{2} + |a|^2 \Re\left\{\frac{i}{a} e^{i(t/2 + t(x,y))}\right\}\right). \quad (20)$$

By comparing (19) and (20) we see that

$$|\xi_2 x - y|^2 - |\xi_1 x - y|^2 = \frac{g(t)}{\sin\frac{t}{2}}\Big\{|\xi_2 x - y|^2 + |\xi_1 x - y|^2\Big\} \geqslant 0. \qquad (21)$$

Since

$$\frac{g(t)}{\sin\frac{t}{2}} = \sqrt{1 - \frac{\sin^2\frac{\alpha}{2}}{\sin^2\frac{t}{2}}} \leqslant \sqrt{1 - \sin^2\frac{\alpha}{2}} = \cos\frac{\alpha}{2},$$

we come to the inequality

$$|\xi_2 x - y|^2 - |\xi_1 x - y|^2 \leqslant \cos\frac{\alpha}{2}\Big\{|\xi_2 x - y|^2 + |\xi_1 x - y|^2\Big\},$$

as needed. ∎

Since $|r_j| = 1,\ |\varphi_n| = |\varphi_n^*|$ on $\mathbb{T}$ and

$$V^{-1}\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{\xi_2 - \xi_1}\begin{bmatrix} \xi_2 x - y \\ -\xi_1 x + y \end{bmatrix},$$

it follows immediately from (18) and Lemma 2 that

$$|u_k| \leqslant |v_k| \leqslant \cot\frac{\alpha}{4}|u_k|, \qquad k = 0, 1, \ldots, l. \qquad (22)$$

Let us now analyze the squeezing effect produced by the matrices A_k, defined in (15). Take

$$A(z,\varepsilon) \stackrel{\text{def}}{=} R^{-1}V^{-1}T(z, a(1+\varepsilon))V = R^{-1}V^{-1}V_\varepsilon R(z,\varepsilon)V_\varepsilon^{-1}V,$$

where V_ε reduces $T(z, a(1+\varepsilon))$ to diagonal form $R(z,\varepsilon)$ (cf. (9)). Under the condition $|1+\varepsilon| = 1$, which is always assumed to hold, we have $r_j(\varepsilon) = r_j,\ z_j(\varepsilon) = z_j$ and $R(z,\varepsilon) = R$.

Next,

$$V_\varepsilon = \begin{pmatrix} 1 & 1 \\ \xi_1(\varepsilon) & \xi_2(\varepsilon) \end{pmatrix}, \qquad V_\varepsilon^{-1} = \frac{1}{\xi_2(\varepsilon) - \xi_1(\varepsilon)}\begin{pmatrix} \xi_2(\varepsilon) & -1 \\ -\xi_1(\varepsilon) & 1 \end{pmatrix}$$

with $a(1+\varepsilon)\xi_j(\varepsilon) = z_j(\varepsilon) - z = z_j - z,\ j = 1, 2$. We have

$$V^{-1}V_\varepsilon = I + \frac{1}{\xi_2 - \xi_1}\begin{pmatrix} \xi_1 - \xi_1(\varepsilon) & \xi_2 - \xi_2(\varepsilon) \\ \xi_1(\varepsilon) - \xi_1 & \xi_2(\varepsilon) - \xi_2 \end{pmatrix},$$

$$V_\varepsilon^{-1}V = I - \frac{1}{\xi_2(\varepsilon) - \xi_1(\varepsilon)}\begin{pmatrix} \xi_1 - \xi_1(\varepsilon) & \xi_2 - \xi_2(\varepsilon) \\ \xi_1(\varepsilon) - \xi_1 & \xi_2(\varepsilon) - \xi_2 \end{pmatrix}.$$

Since

$$\xi_2(\varepsilon) - \xi_1(\varepsilon) = \frac{z_2 - z_1}{a(1+\varepsilon)} = \frac{\xi_2 - \xi_1}{1+\varepsilon}, \qquad \xi_j - \xi_j(\varepsilon) = \frac{z_j - z}{a}\gamma_\varepsilon, \qquad \gamma_\varepsilon = \frac{\varepsilon}{1+\varepsilon},$$

then

$$V^{-1}V_\varepsilon = I + \gamma_\varepsilon B(z), \qquad V_\varepsilon^{-1}V = I - \varepsilon B(z),$$

where

$$B(z) = \frac{1}{z_2 - z_1}\begin{pmatrix} z_1 - z & z_2 - z \\ z - z_1 & z - z_2 \end{pmatrix}.$$

Hence

$$\begin{aligned} A(z,\varepsilon) &= R^{-1}(I + \gamma_\varepsilon B(z))R(I - \varepsilon B(z)) = R^{-1}(R + \gamma_\varepsilon BR - \varepsilon RB - \varepsilon\gamma_\varepsilon BRB) \\ &= R^{-1}(R + \varepsilon BR - \varepsilon RB - \tilde{E}(z,\varepsilon)), \qquad \tilde{E}(z,\varepsilon) = \frac{\varepsilon^2}{1+\varepsilon}BR(I+B). \end{aligned}$$

It is a matter of routine computation (we use $z_1 z_2 = \rho^2 z$ at the last step) to show that

$$R^{-1}BR - R = R^*BR - B = \frac{1}{\rho}\begin{pmatrix} 0 & \frac{z_2 - z}{r_1} \\ \frac{z_1 - z}{r_2} & 0 \end{pmatrix} = \frac{1}{\rho^2}\begin{pmatrix} 0 & -i\eta_2(z) \\ -i\eta_1(z) & 0 \end{pmatrix}$$

with

$$\eta_j(z) \stackrel{\text{def}}{=} \frac{iz_j}{z}(z_j - z) = |\eta_j(z)|e^{i\tau_j(z)}, \qquad j = 1, 2. \tag{23}$$

Note that $|z_j| = \rho < 1$ implies $\eta_j \neq 0$. Finally,

$$A(z,\varepsilon) = I + \delta Q - E(z,\delta); \quad \delta = \frac{\varepsilon}{i\rho^2},\ E(z,\delta) = R^{-1}\tilde{E}(z,\varepsilon), Q = \begin{pmatrix} 0 & \eta_2 \\ \eta_1 & 0 \end{pmatrix}. \tag{24}$$

Recall, that for a vector $h = [x, y]' \in \mathbb{C}^2$ with nonzero entries we defined $t(x, y) = \arg(x\bar{y})$.

LEMMA 3. *Let* $\beta =^{\text{def}} \frac{\pi+\alpha}{2} > \alpha$, *and let a nonzero vector* $h = [x, y]' \in \mathbb{C}^2$ *satisfy*

$$|y| \leqslant |x| \leqslant C_1(a)|y| \tag{25}$$

and

$$\frac{5\pi}{6} < t(x, y) + \tau_1(z) < \frac{7\pi}{6} \tag{26}$$

for some $z \in \Delta_\beta \subset \Delta_\alpha^0$. *Then for small enough* $|\varepsilon| \leqslant \varepsilon_0(a) < 1$ *with* $|1 + \varepsilon| = 1$ *and for all such* z *the inequality*

$$||A(z, \varepsilon)h||^2 \leqslant \left(1 - \frac{|a|}{4\rho C_1(a)}|\varepsilon|\right)||h||^2 \tag{27}$$

holds.

Proof. We have

$$||A(z, \varepsilon)h||^2 = ||h||^2 + 2\Re\{\delta(Qh, h)\} + F(z, \varepsilon, h). \tag{28}$$

Write $\delta = |\delta|e^{i\tau} \neq 0$. Then

$$Qh = \begin{bmatrix} \eta_2 y \\ \eta_1 x \end{bmatrix} = \begin{bmatrix} |\eta_2 y|e^{i(t(y)+\tau_2)} \\ |\eta_1 x|e^{i(t(x)+\tau_1)} \end{bmatrix}$$

and

$$\delta(Qh, h) = |\delta|\,|xy|\left\{|\eta_2|e^{(-t(x,y)+\tau_2+\tau)} + |\eta_1|e^{(t(x,y)+\tau_1+\tau)}\right\},$$

$$\Re\{\delta(Qh, h)\} = |\delta|\,|xy|\{|\eta_2|\cos(t(x, y) - \tau_2 - \tau) + |\eta_1|\cos(t(x, y) + \tau_1 + \tau)\}.$$

By the definition of η_j (23) and (5), (6)

$$\eta_1\eta_2 = -\frac{\rho^2 r_1 r_2 (z - \rho r_1)(z - \rho r_2)}{z^2} = -\frac{\rho^2}{z}\left\{z^2 - (z + 1)z + \rho^2 z\right\} = \rho^2(1 - \rho^2) > 0,$$

so that $|\eta_1\eta_2| = \rho^2|a|^2$ and $\tau_1 + \tau_2 = 0$. Next, the condition $|1 + \varepsilon| = 1$, which is equivalent to $-2\Re\varepsilon = |\varepsilon|^2$ or $2\Im\delta = \rho^2|\delta|^2$, gives

$$2|\delta|\sin\tau = \rho^2|\delta|^2, \qquad \sin\tau = \frac{\rho^2}{2}|\delta| = \frac{|\varepsilon|}{2}.$$

Hence, $\tau = O(\varepsilon)$ as $\varepsilon \to 0$ and in any case $0 < \tau < \pi/6$ for $|\varepsilon| < 1$. In view of (26)

$$\frac{2\pi}{3} < t(x,y) + \tau_1(z) \pm \tau < \frac{4\pi}{3},$$

so that $\cos(t(x,y) + \tau_1(z) \pm \tau) < -1/2$ and

$$\Re\{\delta(Qh,h)\} < -\frac{|\delta|\,|xy|}{2}(|\eta_1| + |\eta_2|) < -|\delta|\,|xy|\,\rho|a|.$$

By (25)

$$|xy| \geqslant \frac{|x|^2 + |y|^2}{2(C_1(a)+1)} = \frac{||h||^2}{2(C_1(a)+1)} \geqslant \frac{||h||^2}{4C_1(a)},$$

which leads to the relation

$$2\Re\{\delta(Qh,h)\} < -\frac{\rho|a|}{2C_1(a)}|\delta|||h||^2 = -\frac{|a|}{2\rho C_1(a)}|\varepsilon|||h||^2.$$

It remains to estimate the last term F in (28). Clearly, $F = O(\varepsilon^2)||h||^2$, and we only have to make more precise the value "O". Since $1 - \rho \leqslant |z_j - z| \leqslant 1 + \rho < 2$, then

$$||Q|| \leqslant 2, \qquad ||B|| \leqslant \frac{4}{|z_1 - z_2|}.$$

But (cf. [6 p. 21]) $|z_1 - z_2| \geqslant 2\pi^{-1}|t - \alpha|$, and hence uniformly for $z \in \Delta_\beta$ we have $|z_1 - z_2| \geqslant (\pi - \alpha)/2$ and $||B|| \leqslant 8(\pi - \alpha)^{-1}$. Therefore, $|F| \leqslant C_2(a)|\varepsilon|^2||h||^2$. The proof is complete. ∎

We want to apply the latter result to $[x,y]' = [u_k, v_k]',\ k \geqslant 1$. Whereas (25) holds by (22) (with $C_1(a) = \cot\frac{\alpha}{4}$), a special choice of z and $\{m_k\}$ is called for (the twisting step of the procedure) to meet a much more delicate inequality (26).

Let $w_j = e^{2\pi i\omega_j} \in \mathbb{T},\ j = 1, 2, \ldots, N$. The points $\{w_j\}$ are called rationally independent if $\{1, \omega_1, \ldots, \omega_N\}$ are rationally independent in the usual sense, i.e.,

$$\sum_{j=1}^{N} k_j\omega_j = k, \qquad k_j, k \in \mathbb{Z} \stackrel{\text{def}}{=} \{0, \pm 1, \ldots\}$$

implies $k_1 = k_2 = \cdots = k_N = k = 0$.

The following result is just a version of the famous Kronecker theorem. Regarding the last statement see [8, Lemma 4].

LEMMA 4. *Let* $\{w_1, w_2, \ldots, w_N\}$ *be rationally independent points on* $\mathbb{T}$. *For each positive* $d > 0$ *there is a number* $m_0 = m_0(N, d) \in \mathbb{N}$ *such that for arbitrary sets* $\{v_j\}$, $\{v_j'\}$ *of points on* $\mathbb{T}$ *the system of inequalities*

$$|w_j^m v_j - v_j'| < d, \qquad j = 1, 2, \ldots, N$$

has a solution m *with* $m \leqslant m_0$.

Let us go back to (16) and put

$$\begin{bmatrix} p_k \\ q_k \end{bmatrix} \stackrel{\text{def}}{=} R^{m_k} \begin{bmatrix} u_{k-1} \\ v_{k-1} \end{bmatrix} = \begin{bmatrix} r_1^{m_k} u_{k-1} \\ r_2^{m_k} v_{k-1} \end{bmatrix}, \qquad \begin{bmatrix} u_k \\ v_k \end{bmatrix} = A_k \begin{bmatrix} p_k \\ q_k \end{bmatrix}, \quad k = 1, 2, \ldots, l.$$

We think of the passage from $[u_{k-1}, v_{k-1}]'$ to $[u_k, v_k]'$ as the kth step of our procedure, which is performed in two half-steps:

from $[u_{k-1}, v_{k-1}]'$ to $[p_k, q_k]'$ (the twisting part);

from $[p_k, q_k]'$ to $[u_k, v_k]'$ (the squeezing part).

We wish to show that the behavior of orthonormal polynomials φ_n, related to some sample sequences, at certain points is under the control.

THEOREM 5. *Let* a *be a nonzero point in* $\mathbb{D}$ *and* $\sin\frac{\alpha}{2} = |a|$. *Suppose that the points* $\{\zeta_1, \zeta_2, \ldots, \zeta_N\}$ *are taken on* Δ_β *with* $2\beta = \pi + \alpha$ *which satisfy*

$$w_j = \frac{r_1(\zeta_j)}{r_2(\zeta_j)} \text{ are rationally independent}, \qquad j = 1, 2, \ldots, N. \tag{29}$$

Then there is a number $0 < \varepsilon_0(a) < 1$ *and a sequence* $\Lambda = \{n_1 < n_2 < \cdots\}$ *with uniformly bounded gaps* $m_k = n_k - n_{k-1} = O(1)$ *as* $k \to \infty$, *such that for each sample sequence* $\{a, \Lambda, \varepsilon_k\}$ *with* $|\varepsilon_k| \leqslant \varepsilon_0(a)$ *the relation*

$$\sum_{s=n_1}^{\infty} |\varphi_s(\zeta_j)|^2 \leqslant C(a, N) \sum_{l=1}^{\infty} \prod_{n=1}^{l} \left(1 - \frac{|a|}{4\rho \cot\frac{\alpha}{4}} |\varepsilon_n|\right), \qquad j = 1, 2, \ldots, N \tag{30}$$

holds.

Proof. Let us begin with the function $f = r_1/r_2$. It is not hard to see from (8) (see also [4, Sect. 2] for more details) that

$$r_{1,2}(e^{it}) = e^{i(\frac{t}{2} \pm \lambda)}, \qquad \cos\lambda \stackrel{\text{def}}{=} \frac{\cos\frac{t}{2}}{\cos\frac{\alpha}{2}}, \quad e^{it} \in \Delta_\alpha,$$

so that λ varies from 0 to π. Hence, $f = e^{2i\lambda}$ is a homeomorphism of Δ_α onto $\mathbb{T}$. Since the set $w_j = f(\zeta_j)$ of rationally independent points is dense on the torus $\mathbb{T}_N$, we can pick up the points $\{\zeta_j\}$ from an arbitrary arc inside Δ_α (say, from Δ_β), to meet (29).

Next, $e^{i(\pi-\tau_1)} = -e^{-i\tau_1}$,

$$e^{it(p_k,q_k)} = e^{it(p_k)} - e^{it(q_k)} = \frac{p_k\bar{q}_k}{|p_k q_k|} = f^{m_k}\frac{u_{k-1}\bar{v}_{k-1}}{|u_{k-1}v_{k-1}|},$$

and we apply Lemma 4 with $w_j = f(\zeta_j)$,

$$v_j = \frac{u_{k-1}(\zeta_j)\overline{v_{k-1}(\zeta_j)}}{|u_{k-1}(\zeta_j)v_{k-1}(\zeta_j)|}, \qquad v_j' = -e^{-i\tau_1(\zeta_j)}, \quad j = 1, 2, \ldots, N,$$

and $d = 1/3$ to ensure (26).

Now Lemma 3 comes into play. As $||[p_k, q_k]'|| = ||[u_{k-1}, v_{k-1}]'||$, we have

$$\left|\left|\begin{bmatrix} u_k(\zeta_j) \\ v_k(\zeta_j) \end{bmatrix}\right|\right|^2 = \left|\left|A(\varepsilon_k)\begin{bmatrix} p_k(\zeta_j) \\ q_k(\zeta_j) \end{bmatrix}\right|\right|^2 \leqslant \left(1 - \frac{|a|}{4\rho\cot\frac{\alpha}{4}}|\varepsilon_k|\right)\left|\left|\begin{bmatrix} u_{k-1}(\zeta_j) \\ v_{k-1}(\zeta_j) \end{bmatrix}\right|\right|^2$$

or

$$|u_k(\zeta_j)|^2 + |v_k(\zeta_j)|^2 \leqslant \prod_{n=1}^{k}\left(1 - \frac{|a|}{4\rho\cot\frac{\alpha}{4}}|\varepsilon_n|\right)(|u_0(\zeta_j)|^2 + |v_0(\zeta_j)|^2$$

with $k = 1, 2, \ldots, l,\ j = 1, 2, \ldots, N$. By (17)

$$\left|\left|\begin{bmatrix} \varphi_s(\zeta_j) \\ \varphi_s^*(\zeta_j) \end{bmatrix}\right|\right|^2 \leqslant ||V||^2\left|\left|\begin{bmatrix} u_l(\zeta_j) \\ v_l(\zeta_j) \end{bmatrix}\right|\right|^2$$

or

$$|\varphi_s(\zeta_j)|^2 \leqslant C(a)\prod_{n=1}^{l}\left(1 - \frac{|a|}{4\rho\cot\frac{\alpha}{4}}|\varepsilon_n|\right).$$

Since by Lemma 4 the gaps m_k are uniformly bounded, we see that

$$\sum_{s=n_l}^{n_{l+1}-1}|\varphi_s(\zeta_j)|^2 \leqslant C(a)m_0(N)\prod_{n=1}^{l}\left(1 - \frac{|a|}{4\rho\cot\frac{\alpha}{4}}|\varepsilon_n|\right),$$

which immediately gives (30). ∎

We are in a position now to prove the main result of the paper.

Proof of Theorem 1. It is clear from (13) that $\sum_{n=0}^{\infty} |\varepsilon_n'| = \infty$, that is, $\{a_n'\} \in \mathscr{B}(a')$. We begin with the choice of a. Consider the function $x(\alpha) \stackrel{\text{def}}{=} \frac{1}{4}\tan\frac{\alpha}{2}\tan\frac{\alpha}{4}$, which is monotonically increasing on $(0, \pi)$ and takes all values between zero and infinity. Pick α from the equality $x(\alpha) = M|a'| + 1$ and put

$$a \stackrel{\text{def}}{=} \sin\frac{\alpha}{2}, \qquad 0 < a < 1.$$

Note that in our notation

$$x(\alpha) = \frac{\sin\frac{\alpha}{2}}{4\cos\frac{\alpha}{2}\cot\frac{\alpha}{4}} = \frac{|a|}{4\rho\cot\frac{\alpha}{4}}. \tag{31}$$

Given $N \in \mathbb{N}$, pick the points $\{\zeta_1, \zeta_2, \ldots, \zeta_N\}$ in Δ_β to meet (29). By Theorem 5 we find the number $\varepsilon_0(a)$ and the sequence $\Lambda \subset \mathbb{N}$ such that (30) holds. In view of (13), (31) and $x(\alpha) = M|a'| + 1$, the right-hand side of (30) is finite, which means that

$$\sum_{s=0}^{\infty} |\varphi_s(\zeta_p)|^2 < \infty.$$

It is well known (cf. [11, pp. 45–46; 2, Sect. 20]) that the latter inequality guarantees the existence of masses for the orthogonality measure μ at the points ζ_p.

It remains only to determine $\{\varepsilon_n\}$. Put

$$\varepsilon_k'' = 0, \quad k \notin \Lambda, \qquad |\varepsilon_{n_l}''| = \sum_{j=n_{l-1}}^{n_l - 1} |\varepsilon_j'|, \qquad l \in \mathbb{N}, \quad n_0 = 1.$$

We have $\lim_{n\to\infty} \varepsilon_n'' = 0$ because of the boundedness of the gaps m_k. Let $n_q \leqslant s < n_{q+1}$. Then

$$\sum_{k=1}^{s} |\varepsilon_k''| = \sum_{l=1}^{q} |\varepsilon_{n_l}''| = \sum_{l=1}^{q} \sum_{j=n_{l-1}}^{n_l - 1} |\varepsilon_j'| \leqslant \sum_{j=1}^{s} |\varepsilon_j'|,$$

so that

$$1 \geqslant \frac{\sum_{j=1}^{s} |\varepsilon_j''|}{\sum_{j=1}^{s} |\varepsilon_j'|} \geqslant \frac{\sum_{j=1}^{n_q} |\varepsilon_j''|}{\sum_{j=1}^{n_{q+1}} |\varepsilon_j'|} = \frac{\sum_{j=1}^{n_q - 1} |\varepsilon_j'|}{\sum_{j=1}^{n_{q+1}} |\varepsilon_j'|} = 1 - \frac{\sum_{j=n_q}^{n_{q+1}} |\varepsilon_j'|}{\sum_{j=1}^{n_{q+1}} |\varepsilon_j'|} \to 1.$$

Since $\varepsilon_n'' \to 0$, then $|\varepsilon_{n_l}''| < \varepsilon_0(a)$ for $l \geqslant l_0 + 1$. Define

$$\varepsilon_k = 0, \quad k \notin \Lambda, \qquad |\varepsilon_{n_l}| = \begin{cases} 0 & \text{for } l = 1, 2, \ldots, l_0, \\ |\varepsilon_{n_l}''| & \text{for } l \geqslant l_0 + 1, \end{cases}$$

and $|1+\varepsilon_n| = 1$ for all n. The triple $(a, \Lambda, \{\varepsilon_n\})$ provides the sample sequence equivalent to the original one and $\mu\{\zeta_p\} > 0$, $p = 1, 2, \ldots, N$, as stated. ∎

Remark. As in [9], the following result can be obtained.

Let $\Omega_n \to +\infty$, $n \to \infty$, arbitrarily slow. There exists a sample sequence $(a, \Lambda, \{\varepsilon_n\})$ such that $|a_n - a| = |a\varepsilon_n| \leqslant \Omega_n/n$ and the corresponding orthogonality measure μ has infinitely many mass points on Δ_α. As a matter of fact, the set of mass points can be taken to be dense on Δ_α.

Let us mention the recent paper [10], where the problem of addition of a finite number of mass points to an absolutely continuous measure with asymptotically periodic reflection coefficients is studied. It is proved in [10, Theorem 3] that the difference of the corresponding reflection coefficients goes to zero in this case.

3. MASS POINTS ON THE WHOLE CIRCLE AND SINGULAR CONTINUOUS MEASURES

The argument here relies upon the equivalence

$$\mu\{\zeta\} > 0, \qquad \zeta \in \mathbb{T} \Leftrightarrow \sum_{n=0}^{\infty} |\varphi_n(\zeta)|^2 < \infty \tag{32}$$

mentioned above. Thereby, to prove that $\mu\{\zeta\} > 0$ ($\mu\{\zeta\} = 0$) we need certain upper (lower) bounds for the orthonormal polynomials.

We begin with the basic Szegő recurrences for monic orthogonal polynomials on the unit circle

$$\Phi_n^*(z) = \Phi_{n-1}^*(z) + \bar{a}_n z \Phi_{n-1}(z) = \Phi_{n-1}^*(z)\left(1 + \bar{a}_n z \frac{\Phi_{n-1}(z)}{\Phi_{n-1}^*(z)}\right), \qquad \Phi_n = \frac{\varphi_n}{\kappa_n}.$$

(cf. [1 formula (8.1)]). Iteration of the latter leads to

$$\Phi_n^*(z) = \prod_{k=1}^{n}(1 + \bar{a}_k z b_{k-1}(z)), \qquad b_j = \frac{\Phi_j}{\Phi_j^*}. \tag{33}$$

Since $|b_j| = 1$ on $\mathbb{T}$ and $\Phi_j^* \neq 0$ in $\bar{\mathbb{D}}$, then $|b_j| \leqslant 1$ in $\bar{\mathbb{D}}$ and

$$|\Phi_n^*(z)| = \frac{|\varphi_n^*(z)|}{\kappa_n} \leqslant \prod_{k=1}^{n}(1 + |a_k|), \tag{34}$$

$$|\varphi_n^*(\zeta)| = |\varphi_n(\zeta)| \leqslant \kappa_n \exp\left\{ \sum_{k=1}^{n} |a_k| \right\}, \qquad \zeta \in \mathbb{T}. \tag{35}$$

Recall that the Szegő class of measures on $\mathbb{T}$ is characterized by the inequality $\sum_{n\geqslant 1} |a_n|^2 < \infty$ [1, Theorem 8.2].

THEOREM 6. *Let μ belong to the Szegő class and let its reflection coefficients satisfy*

$$\sum_{n=1}^{\infty} \exp\left\{ -2 \sum_{k=1}^{n} |a_k| \right\} = \infty. \tag{36}$$

Then $\mu\{\zeta\} = 0$ for all $\zeta \in \mathbb{T}$, i.e., μ is a continuous measure. Conversely, let $a_n < 0$ and assume that series (36) *converges. Then $\mu\{1\} > 0$.*

Proof. We invoke the second kind polynomials ψ_n (compared to the first kind polynomials $\varphi_n = \varphi_n(\{a_n\})$), which are defined by the sequence of reflection coefficients $\{-a_n\}$. The relation between φ_n and ψ_n is given by

$$\varphi_n^*(z)\psi_n(z) + \varphi_n(z)\psi_n^*(z) = 2z^n$$

(cf. [1, formula (1.17)]), which for $z = \zeta \in \mathbb{T}$ is

$$\overline{\varphi_n(\zeta)}\psi_n(\zeta) + \varphi_n(\zeta)\overline{\psi_n(\zeta)} = 2\Re\left\{ \varphi_n(\zeta)\overline{\psi_n(\zeta)} \right\} = 2. \tag{37}$$

Hence $|\varphi_n\psi_n| \geqslant 1$ on the circle, and the upper bound for ψ_n yields the lower bound for φ_n.

The general formula for κ_n (cf. [1, formula (8.6)])

$$\kappa_n^{-2} = \prod_{k=1}^{n} (1 - |a_k|^2) \tag{38}$$

shows that φ_n and ψ_n have the same leading coefficients. Moreover, within the Szegő class $\kappa_n^2 \nearrow \kappa^2 < \infty$. It follows now from (35) applied to ψ_n that

$$|\varphi_n(\zeta)|^2 \geqslant |\psi_n(\zeta)|^{-2} \geqslant \kappa^{-2} \exp\left\{ -2 \sum_{k=1}^{n} |a_k| \right\}.$$

By (36) the series in (32) diverges, which implies the first statement of the theorem.

Suppose now that $a_n = \bar{a}_n$. Then $\varphi_n,\ \psi_n$ are real on the real line and by (33) and (38)

$$\Phi_n(1) = \Phi_n^*(1) = \prod_{k=1}^{n}(1 + a_n), \qquad \varphi_n^2(1) = \kappa_n^2 \Phi_n^2(1) = \prod_{k=1}^{n} \frac{1 + a_k}{1 - a_k}.$$

If in addition $a_n < 0$, then

$$\psi_n^2(1) = \prod_{k=1}^{n} \frac{1 + |a_k|}{1 - |a_k|} = \exp\left\{\sum_{k=1}^{n} \log \frac{1 + |a_k|}{1 - |a_k|}\right\}.$$

An elementary inequality $\log \frac{1+x}{1-x} \geqslant 2x,\ 0 \leqslant x < 1$, gives

$$\psi_n^2(1) \geqslant \exp\left\{2 \sum_{k=1}^{n} |a_k|\right\} = \exp\left\{-2 \sum_{k=1}^{n} a_k\right\}.$$

From (37) we see that $\varphi_n(1)\psi_n(1) = 1$ and

$$\varphi_n^2(1) \leqslant \exp\left\{2 \sum_{k=1}^{n} a_k\right\}, \qquad \sum_{n=1}^{\infty} \varphi_n^2(1) \leqslant \sum_{n=1}^{\infty} \exp\left\{2 \sum_{k=1}^{n} a_k\right\}.$$

The second statement of the theorem follows immediately from (32). The proof is complete. ∎

As a direct consequence we obtain the following result (cf. [3, Theorem VIII]).

COROLLARY 7. *If* $|a_n| \leqslant (2n)^{-1}$ *for* $n \geqslant n_0$ *then* $\mu\{\zeta\} = 0$ *for all* $\zeta \in \mathbb{T}$. *If* $a_n < 0$ *and* $|a_n| \geqslant (1/2 + \varepsilon)n^{-1}$ *for* $\varepsilon > 0$ *and* $n \geqslant n_0$, *then* $\mu\{1\} > 0$.

There is yet another upper bound for general orthonormal polynomials valid beyond the Szegő class (cf. [3, Theorem III])

$$|\varphi_n(\zeta)| \leqslant \exp\left\{\frac{1}{1 - \gamma^2} \sum_{k=1}^{n} |a_k|\right\}, \qquad \zeta \in \mathbb{T}, \quad \gamma \stackrel{\text{def}}{=} \sup_n |a_n| < 1, \qquad (39)$$

which provides a number of curious examples of singular continuous measures given by their reflection coefficients.

EXAMPLE. Take a set $\Lambda = \{n_1 < n_2 < \cdots\}$ of positive integers with $m_k = n_{k+1} - n_k \to \infty$ as $k \to \infty$. We call a sequence $\{a_n\}$ of complex numbers *lacunary* if $a_n = 0,\ n \notin \Lambda$.[2]

[2] The term "sparse" is used in the theory of Schrödinger operators.

Consider a lacunary sequence of reflection coefficients such that $a_{n_k} = a,\ 0 < |a| < 1$. We have

$$\sum_{k=1}^{n} |a_k| = j|a|, \quad n_j \leqslant n < n_{j+1} - 1, \qquad j \geqslant 1.$$

Let ψ_n be the corresponding second kind polynomials. By (39) and $|\varphi_n \psi_n| \geqslant 1$ on $\mathbb{T}$ we see that

$$|\varphi_n(\zeta)|^2 \geqslant \exp\left\{ -\frac{2}{1-\gamma^2} \sum_{k=1}^{n} |a_k| \right\}.$$

Therefore,

$$\sum_{n=n_1}^{\infty} |\varphi_n(\zeta)|^2 \geqslant \sum_{j=1}^{\infty} \sum_{n=n_j}^{n_{j+1}-1} \exp\left\{ -\frac{2j|a|}{1-|a|^2} \right\} = \sum_{j=1}^{\infty} (n_{j+1} - n_j) \exp\left\{ -\frac{2j|a|}{1-|a|^2} \right\}.$$

Assume now that the gaps m_j grow exponentially fast, more precisely,

$$\log m_j \geqslant 2j|a|(1 - |a|^2)^{-1}.$$

Then the latter series diverges, and by (32) $\mu\{\zeta\} = 0$ for all $\zeta \in \mathbb{T}$, that is, the measure is continuous. It remains only to refer to Khrushchev's theorem [7, Corollary 9.2] which states that such measures are singular.

The more general type of examples can be easily manufactured. Let

$$a_{n_k} = \gamma_k, \qquad 0 < \limsup_{k \to \infty} |\gamma_k| < 1. \tag{40}$$

Then the measure generated by such a sequence of reflection coefficients is singular continuous as long as the gaps m_j grow fast enough.

On the other hand, let $\sum_k |\gamma_k|^2 < \infty$ in (40), that leads to a subclass of measures in the Szegő class with the lacunary reflection coefficients. We have as above

$$\sum_{n=n_1}^{\infty} \exp\left\{ -2 \sum_{k=1}^{n} |a_k| \right\} = \sum_{j=1}^{\infty} (n_{j+1} - n_j) \exp\left\{ -2 \sum_{k=1}^{j} |\gamma_k| \right\}.$$

By Schwarz's inequality

$$\left(\sum_{k=1}^{j} |\gamma_k| \right)^2 \leqslant j \sum_{k=1}^{j} |\gamma_k|^2 \leqslant C^2 j, \qquad C^2 \stackrel{\text{def}}{=} \sum_{k=1}^{\infty} |\gamma_k|^2,$$

so that

$$\sum_{j=1}^{\infty}(n_{j+1}-n_j)\exp\left\{-2\sum_{k=1}^{j}|\gamma_k|\right\} \geqslant \sum_{j=1}^{\infty}(n_{j+1}-n_j)e^{-2C\sqrt{j}}.$$

By Theorem 6 the corresponding measure is continuous as long as the gaps m_j grow exponentially fast. It is not clear though whether all such measures are pure absolutely continuous.

REFERENCES

1. L. Ya. (aka Ya. L.) Geronimus "Orthogonal Polynomials," Consultants Bureau, New York, 1961.
2. Ya. L. Geronimus, "Polynomials Orthogonal on a Circle and their Applications," Series and Approximations, American Mathematical Society Translations No. 1, Vol. 3, pp. 1–78, Providence, RI, 1962.
3. Ya. L. Geronimus, On certain estimates for orthogonal polynomials depending on parameters, *Anal. Math.* **3** (1977), 95–108.
4. L. Golinskii, Geronimus polynomials and weak convergence on a circular arc, *Methods Appl. Anal.* **6** (1999), 421–436.
5. L. Golinskii and P. Nevai, Szegő difference equations, transfer matrices and orthogonal polynomials on the unit circle, *Comm. Math. Phys.* **223** (2001), 223–259.
6. L. Golinskii, P. Nevai, F. Pinter, and W. Van Assche, Perturbation of orthogonal polynomials on an arc of the unit circle, II, *J. Approx. Theory* **96** (1999), 1–32.
7. S. Khrushchev, Schur's algorithm, orthogonal polynomials and convergence of Wall's continued fractions in $L^2(\mathbb{T})$, *J. Approx. Theory* **108** (2001), 161–248.
8. S. N. Naboko, On the dense point spectrum of Schrödinger and Dirac operators, *Theory Math. Phys.* **68** (1986), 18–28.
9. S. N. Naboko and S. I. Yakovlev, The discrete Schrödinger operator. The point spectrum lying on the continuous spectrum, *St. Petersburg Math. J.* **4** (1993), 559–568.
10. F. Peherstorfer and R. Steinbauer, Orthogonal polynomials on the circumference and arcs of the circumference, *J. Approx. Theory* **102** (2000) 96–119.
11. J. A. Shohat and J. D. Tamarkin, The problem of moments, "Mathematical Surveys," Vol. 1, Amer. Math. Soc., Providence, RI, 1970.
12. G. Szegő, Orthogonal polynomials, "*American Mathematical Society Colloquium Publications*," 4th ed., Vol. 23, Amer. Math. Soc. Providence, RI, 1975.